CAMBRIDGE ASSESSMENT

Sixth Term Examination Papers **9475**

MATHEMATICS 3 Afternoon

WEDNESDAY 22 JUNE 2011 Time: 3 hours

Additional Materials: Answer Booklet
Formulae Booklet

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 8 printed pages and 4 blank pages.

[Turn over

Section A: Pure Mathematics

1 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right) u = 0\,.$$

(ii) Show that substituting $y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

2 The polynomial $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,,$$

where $n \geqslant 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $\mathrm{f}(x) = 0$ has a rational root p/q, where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}\mathrm{f}(p/q)$, find the value of q and deduce that any rational root of the equation $\mathrm{f}(x) = 0$ must be an integer.

(i) Show that the nth root of 2 is irrational for $n \geqslant 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geqslant 2$.

3 Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad\qquad (p \neq 0,\ q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3\,,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β.

Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2 - 2^{\frac{1}{3}})}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$.

Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r.

4 The following result applies to any function f which is continuous, has positive gradient and satisfies $\mathrm{f}(0) = 0$:

$$ab \leqslant \int_0^a \mathrm{f}(x)\,\mathrm{d}x + \int_0^b \mathrm{f}^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geqslant 0$ and $b \geqslant 0$.

(i) By considering the graph of $y = \mathrm{f}(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $\mathrm{f}(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leqslant a \leqslant \frac{1}{2}\pi$ and $0 \leqslant b \leqslant 1$,

$$ab \leqslant b \arcsin b + \sqrt{1 - b^2} - \cos a\,.$$

Deduce that, for $t \geqslant 1$,

$$\arcsin(t^{-1}) \geqslant t - \sqrt{t^2 - 1}\,.$$

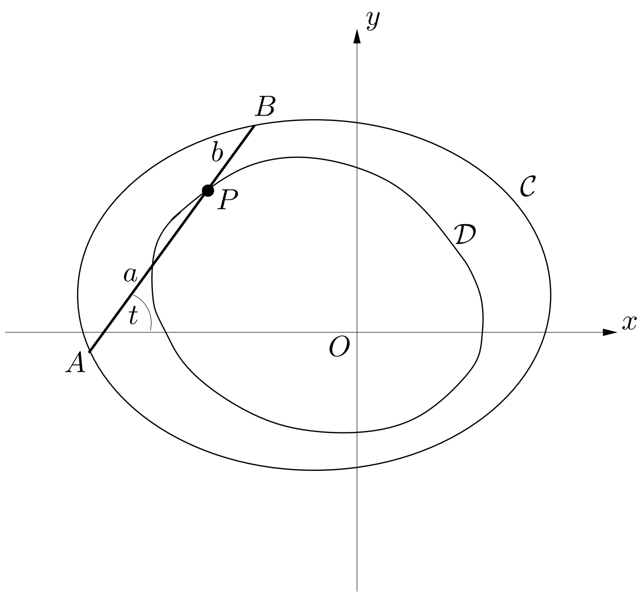
5 A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2 \, \mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t \,. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right)\sin t \right) \mathrm{d}t \,.$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

6 The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t\,, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\,\mathrm{d}v\,, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x\,.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

7 Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a}\right)^n\,,$$

where n is a positive integer and a is any given positive integer.

(i) In the case when n is even, show by induction that T_n can be written in the form

$$A_n + B_n\sqrt{a(a+1)}\,,$$

where A_n and B_n are integers (depending on a and n) and $A_n^2 = a(a+1)B_n^2 + 1$.

(ii) In the case when n is odd, show by considering $(\sqrt{a+1} + \sqrt{a})T_m$ where m is even, or otherwise, that T_n can be written in the form

$$C_n\sqrt{a+1} + D_n\sqrt{a}\,,$$

where C_n and D_n are integers (depending on a and n) and $(a+1)C_n^2 = aD_n^2 + 1$.

(iii) Deduce that, for each n, T_n can be written as the sum of the square roots of two consecutive integers.

8 The complex numbers z and w are related by

$$w = \frac{1 + \mathrm{i}z}{\mathrm{i} + z}\,.$$

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real. Express u and v in terms of x and y.

(i) By setting $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$ with one point omitted.

(ii) Find the locus of w when the locus of z is the line segment $y = 0$, $-1 < x < 1$.

(iii) Find the locus of w when the locus of z is the line segment $x = 0$, $-1 < y < 1$.

(iv) Find the locus of w when the locus of z is the line $y = 1$, $-\infty < x < \infty$.

Section B: Mechanics

9 Particles P and Q have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius a which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at O. Initially, the particles are in equilibrium.

Equilibrium is slightly disturbed and Q begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle θ between OQ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g\,.$$

(i) Given that Q loses contact with the cylinder first, show that it does so when $\theta = \beta$, where β satisfies

$$15\cos\beta + 6\sin\beta = 10.$$

(ii) Show also that while P and Q are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin\theta + \cos\theta)$.

10 Particles P and Q, each of mass m, lie initially at rest a distance a apart on a smooth horizontal plane. They are connected by a light elastic string of natural length a and modulus of elasticity $\frac{1}{2}ma\omega^2$, where ω is a constant.

Then P receives an impulse which gives it a velocity u directly away from Q. Show that when the string next returns to length a, the particles have travelled a distance $\frac{1}{2}\pi u/\omega$, and find the speed of each particle.

Find also the total time between the impulse and the subsequent collision of the particles.

11 A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin \phi = 2a \sin \tfrac{1}{2}\theta \, .$$

Show further that the magnitude of the couple is

$$\frac{mga^2 \sin \theta}{\sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}} \, .$$

The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta} \, .$$

[Turn over

Section C: Probability and Statistics

12 The random variable N takes positive integer values and has pgf (probability generating function) $\mathrm{G}(t)$. The random variables X_i, where $i = 1, 2, 3, \ldots$, are independently and identically distributed, each with pgf $\mathrm{H}(t)$. The random variables X_i are also independent of N. The random variable Y is defined by

$$Y = \sum_{i=1}^{N} X_i \, .$$

Given that the pgf of Y is $\mathrm{G}(\mathrm{H}(t))$, show that

$$\mathrm{E}(Y) = \mathrm{E}(N)\mathrm{E}(X_i) \qquad \text{and} \qquad \mathrm{Var}\,(Y) = \mathrm{Var}\,(N)\big(\mathrm{E}(X_i)\big)^2 + \mathrm{E}(N)\mathrm{Var}\,(X_i)\, .$$

A fair coin is tossed until a head occurs. The total number of tosses is N. The coin is then tossed a further N times and the total number of heads in these N tosses is Y. Find in this particular case the pgf of Y, $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

13 In this question, the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, so for example $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

(i) A bag contains n balls, of which b are black. A sample of k balls is drawn, one after another, at random *with* replacement. The random variable X denotes the number of black balls in the sample. By considering

$$\frac{\mathrm{P}(X = r+1)}{\mathrm{P}(X = r)}\,,$$

show that, in the case that it is unique, the most probable number of black balls in the sample is

$$\left\lfloor \frac{(k+1)b}{n} \right\rfloor .$$

Under what circumstances is the answer not unique?

(ii) A bag contains n balls, of which b are black. A sample of k balls (where $k \leqslant b$) is drawn, one after another, at random *without* replacement. Find, in the case that it is unique, the most probable number of black balls in the sample.

Under what circumstances is the answer not unique?

BLANK PAGE

BLANK PAGE

BLANK PAGE

BLANK PAGE